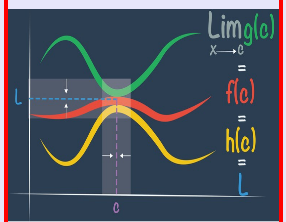
Math 261

Spring 2022

Lecture 17



Critical Numbers within the domain of $f(x)$

are number c such that

1) c is in the domain of $f(x)$.

2) $f'(c) = 0$, or $f'(c)$ does not exist.

Ex: $f(x) = x^3 - 3x^2 + 1$

Polynomial function $\Rightarrow$ Domain: $(-\infty, \infty)$

$f'(x) = 3x^2 - 6x$

$f'(x) = 0 \Rightarrow 3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0$

$3x = 0 \qquad x - 2 = 0$

$\boxed{x=0} \qquad \boxed{x=2}$

Critical Numbers

$f'(x)$ is also a Polynomial

$\Rightarrow$ It is defined everywhere.

$(0, f(0)), (2, f(2))$

Critical Points

Consider $f(x) = \sqrt[5]{x^3}(4-x)$

odd root $\Rightarrow$ No restriction

Domain: $(-\infty, \infty)$

1) Find $f'(x)$

$f(x) = x^{\frac{3}{5}}(4-x)$

$f'(x) = \dfrac{12-8x}{5x^{2/5}}$

$f'(x) = 0 \rightarrow 12-8x=0 \Rightarrow x = \frac{3}{2}$

$f'(x)$ is undefined $\rightarrow 5x^{2/5} = 0 \rightarrow x = 0$

$$f'(x) = \frac{3}{5}x^{\frac{3}{5}-1}(4-x) + x^{\frac{3}{5}}(-1)$$
$$= \frac{3}{5}x^{\frac{-2}{5}}(4-x) - x^{\frac{3}{5}}$$
$$= x^{\frac{-2}{5}}\left[\frac{3}{5}(4-x) - x\right]$$
$$= x^{\frac{-2}{5}}\left[\frac{12}{5} - \frac{3}{5}x - x\right]$$
$$= x^{\frac{-2}{5}}\left[\frac{12}{5} - \frac{8}{5}x\right]$$
$$= \frac{1}{5}x^{\frac{-2}{5}}(12-8x)$$

C.N. $\Rightarrow x = 0, \quad x = \frac{3}{2}$

Critical Points $\Rightarrow (0, f(0)), \left(\frac{3}{2}, f\left(\frac{3}{2}\right)\right)$

Consider $f(x) = \dfrac{x-1}{x^2+4}$

1) Domain: $(-\infty, \infty)$ because $x^2+4 \neq 0$

2) $f'(x) = \dfrac{1(x^2+4)-(x-1)\cdot 2x}{(x^2+4)^2} = \dfrac{x^2+4-2x^2+2x}{(x^2+4)^2}$

$$= \frac{-x^2+2x+4}{(x^2+4)^2}$$

3) Critical Numbers:

$f'(x)=0$ or $f'(x)$ undefined $\rightarrow (x^2+4)^2 = 0$

Has no Solution

$-x^2+2x+4=0$

$x^2-2x-4=0$

$x^2-2x+1 = 4+1$

$(x-1)^2 = 5 \qquad x-1 = \pm\sqrt{5} \qquad x = 1 \pm \sqrt{5}$

C.N. $1+\sqrt{5}, 1-\sqrt{5}$

Critical Points $\Rightarrow (1+\sqrt{5}, f(1+\sqrt{5})), (1-\sqrt{5}, f(1-\sqrt{5}))$

Find all Critical numbers of $f(x) = 2\cos x + \sin^2 x$ on $[0, 2\pi]$.

$\hookrightarrow f'(x) = 0$, $f'(x)$ undefined

$f'(x) = -2\sin x + 2\sin x \cos x$

$f'(x)$ is defined everywhere

$f'(x) = 0 \Rightarrow -2\sin x + 2\sin x \cos x = 0$

$2\sin x \cos x - 2\sin x = 0$

$2\sin x\,[\cos x - 1] = 0$

$\sin x = 0$ $\qquad$ $\cos x = 1$

$x = 0, \pi$ $\qquad$ $x = 0, 2\pi$

C.N.: $0, \pi, 2\pi$

Critical Points: $(0, f(0)), (\pi, f(\pi)), (2\pi, f(2\pi))$

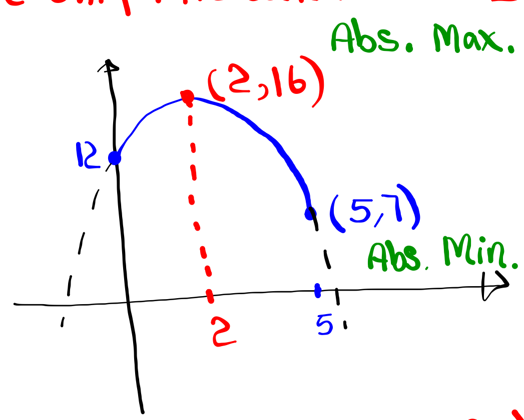
Consider $f(x) = 12 + 4x - x^2$ on $[0, 5]$

1) Domain $(-\infty, \infty)$ we are only intereseted on $[0,5]$

2) $f(0) = 12$

3) $f(5) = 7$

4) $f(x)$ is a Parabola open downward



5) $f'(x) = 4 - 2x$

6) C.N. $f'(x) = 0$

$4 - 2x = 0$

$\boxed{x = 2}$

7) C.P. $(2, f(2))$

$\Rightarrow (2, 16)$

$f(x) = 2x^3 - 3x^2 - 12x + 1$ on $[-2, 3]$

$f(0) = 1$

1) $f(-2) = -3$

2) $f(3) = -8$

3) $f'(x) = 6x^2 - 6x - 12$

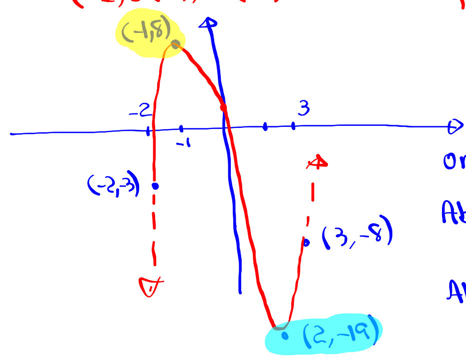
4) C.N. $f'(x) = 0$ $\quad 6x^2 - 6x - 12 = 0 \quad x^2 - x - 2 = 0$

$(x-2)(x+1) = 0 \quad \boxed{x=2} \quad \boxed{x=-1}$

5) C.P. $(2, f(2)) = (2, -19)$

$(-1, f(-1)) = (-1, 8)$

$f(x)$ is a Polynomial.
Cont. everywhere



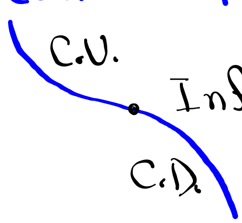
on $[-2, 3]$

Abs. Max at $(-1, 8)$

Abs. Min at $(2, -19)$

Inflection Points:

Those points on the graph within the domain of $f(x)$ where concavity changes.



How to find possible inflection Points:

$f''(x) = 0$ OR $f''(x)$ is undefined

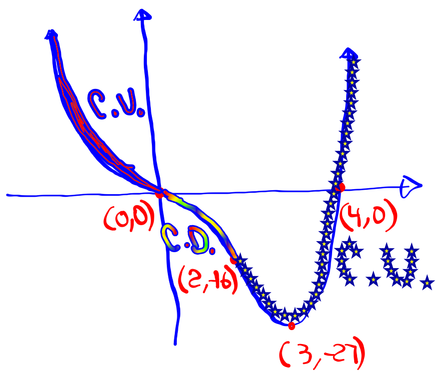
Consider $f(x) = x^4 - 4x^3 = x^3(x-4)$

1) All intercepts:

Y-Int $(0,0)$ x-Int $(0,0), (4,0)$

2) $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$ C.N. $x=0, x=3$

3) $f''(x) = 12x^2 - 24x = 12x(x-2)$ P.I.P. at $x=0, x=2$



Concavity changed at $x=0$ and $x=2$

Inflection Points are

$(0, f(0)) = (0,0)$

$(2, f(2)) = (2,-16)$

$f(x) = x^{\frac{2}{3}}(6-x)^{\frac{1}{3}}$

index = 3

$f(x) = \sqrt[3]{x^2}\ \sqrt[3]{6-x}$

odd root → No restrictions

Domain $(-\infty, \infty)$

1) x-Int $(0,0), (6,0)$ 2) Y-Int $(0,0)$

3) $f'(x) = \dfrac{4-x}{x^{1/3}(6-x)^{2/3}}$ $f''(x) = \dfrac{-8}{x^{4/3}(6-x)^{5/3}}$

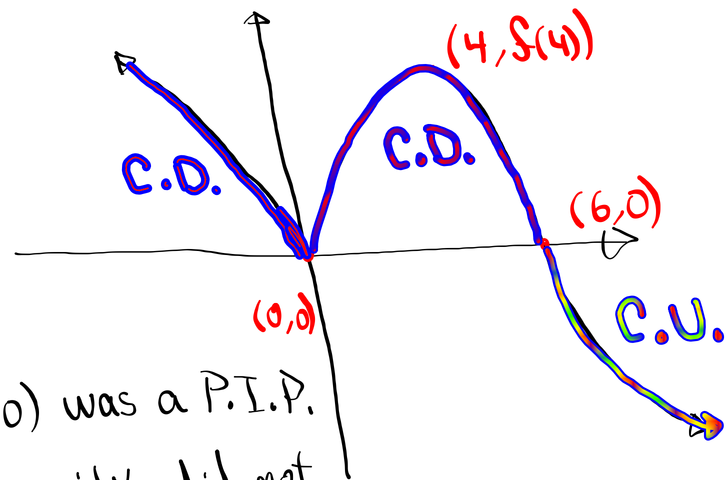
C.N. $f'(x) = 0 \rightarrow x=4$

$f'(x)$ undefined $\rightarrow x=0,\ x=6$

P.I.P.

$f''(x) = 0$, $f''(x)$ undefined

NONE $x=0,\ x=6$



(0,0) was a P.I.P.
Concavity did not change
So (0,0) is not
an inflection Point

(6,0) was a P.I.P., and
Concavity changed.
Since concavity changed,
(6,0) is an I.P.

Given $f(x) = x\sqrt{6-x}$

1) Domain

Defined everywhere

$6-x \geq 0$ why? Index=2 even index

$-x \geq -6$

$x \leq 6 \Rightarrow \boxed{(-\infty, 6]}$

2) Y-Int (0,0)

3) X-Ints $f(x)=0$ $x\sqrt{6-x}=0$ $x=0$, $x=6$

(0,0), (6,0)

4) Find $f'(x) = 1(6-x)^{\frac{1}{2}} + x \cdot \frac{1}{2}(6-x)^{-1/2} \cdot (-1)$

$f(x) = x(6-x)^{1/2}$

$f'(x) = (6-x)^{1/2} - \frac{x}{2}(6-x)^{-1/2}$

$f'(x) = (6-x)^{-1/2}\left[(6-x)^{1} - \frac{x}{2}\right]$

$f'(x) = (6-x)^{-1/2}\left(6-x-\frac{x}{2}\right)$

C.N.

$f'(x)=0 \rightarrow x=4$

$f'(x)$ undefined $\rightarrow x=6$

$f'(x) = (6-x)^{-1/2}\left(6-\frac{3}{2}x\right)$

$f'(x) = \frac{1}{2}(6-x)^{-1/2}(12-3x)$

$$\boxed{f'(x) = \frac{3(4-x)}{2\sqrt{6-x}}}$$

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$$\boxed{f'(x)=\frac{3(4-x)}{2\sqrt{6-x}}}$$

$$\frac{d}{dx}\left[\sqrt{6-x}\right]=\frac{d}{dx}\left[(6-x)^{1/2}\right]$$

$$\frac{1}{2}(6-x)^{-1/2}\cdot(-1)=\frac{-1}{2\sqrt{6-x}}$$

$$f''(x)=\frac{3}{2}\cdot\frac{-1\cdot\sqrt{6-x}-(4-x)\cdot\frac{-1}{2\sqrt{6-x}}}{\left(\sqrt{6-x}\right)^2}$$

$$=\frac{3}{2}\cdot\frac{-\sqrt{6-x}+\frac{4-x}{2\sqrt{6-x}}}{6-x}$$

Multiply $2\sqrt{6-x}$

$$=\frac{3}{2}\cdot\frac{-2(6-x)+4-x}{(6-x)\cdot 2\sqrt{6-x}}$$

$$f''(x)=\frac{3[-12+2x+4-x]}{4(6-x)\sqrt{6-x}}=\frac{3(x-8)}{4(6-x)\sqrt{6-x}}$$

P.I.P.

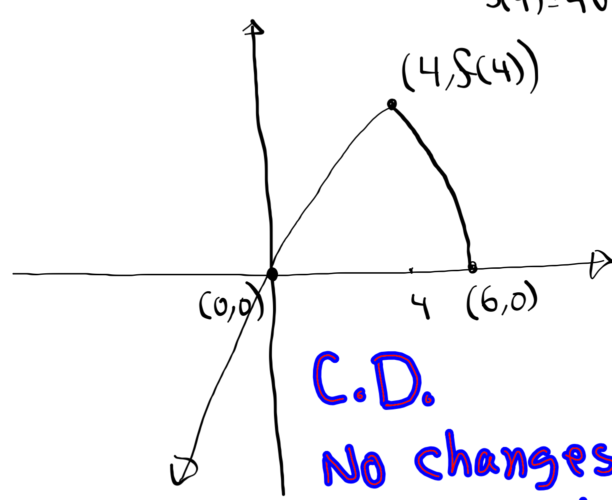
$f''(x)=0$ → at $x=8$ → 8 is not in the domain.

$f''(x)$ undefined → at $x=6$

Rough graph

$f(x)=x\sqrt{6-x}$

$f(4)=4\sqrt{2}$



C.D.

No changes in Concavity

No inflection Point.

$f(x) = \dfrac{x^2}{x^2+3}$

1) Domain $\Rightarrow (-\infty, \infty)$

2) x-Int , Y-Int $(0,0)$

3) $f(-x) = \dfrac{(-x)^2}{(-x)^2+3} = \dfrac{x^2}{x^2+3} = f(x)$ even function Symmetric about Y-axis.

4) $f'(x) = \dfrac{2x(x^2+3) - x^2 \cdot 2x}{(x^2+3)^2} = \dfrac{2x(\cancel{x^2}+3-\cancel{x^2})}{(x^2+3)^2} = \dfrac{6x}{(x^2+3)^2}$

5) C.N. $\rightarrow f'(x)=0$ $6x=0$ $\boxed{x=0}$

$f'(x)$ undefined (None)

6) $f''(x) = 6 \cdot \dfrac{1(x^2+3)^2 - x \cdot 2(x^2+3)^1 \cdot 2x}{[(x^2+3)^2]^2} = 6 \cdot \dfrac{\cancel{(x^2+3)}[x^2+3-4x^2]}{(x^2+3)^{\cancel{4}}}$

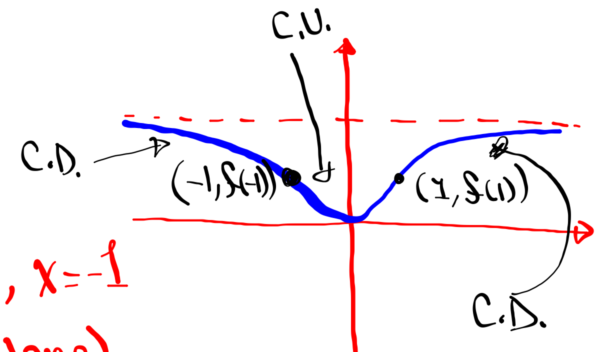
$= \dfrac{6(3-3x^2)}{(x^2+3)^3} = \dfrac{18(1-x^2)}{(x^2+3)^3} = \dfrac{18(1+x)(1-x)}{(x^2+3)^3}$

$f''(x) = \dfrac{18(1+x)(1-x)}{(x^2+3)^3}$

P.I.P.

$f''(x) = 0$ at $x=1$, $x=-1$

$f''(x)$ undefined (None)



I.P. at $(-1, f(-1)) = (-1, \frac{1}{4})$

$(1, f(1)) = (1, \frac{1}{4})$

Find two numbers that have a difference of 20, and a minimum product.

ex: $21 - 1 = 20$ $\quad 21 \cdot 1 = 21$

$32 - 12 = 20$ $\quad 32 \cdot 12 = 384$

$45 - 25 = 20$

$5 - (-15) = 20$ $\quad 5(-15) = -75$

⋮

$x - y = 20$

Solve for y

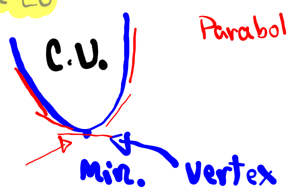
$y = x - 20$

$x \cdot y$ → to be minimum

$x(x-20)$ → to be minimum

$f(x) = x^2 - 20x$

Parabola → open upward



$f'(x) = 0$

$2x - 20 = 0$

$x = 10$

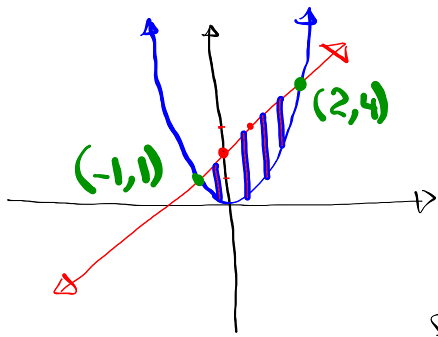
$\boxed{x = 10}$ → $y = 10 - 20$ → $\boxed{y = -10}$

$f''(x) = 2$

C.U. → Min. Point

10 & −10.

Graph $y = x + 2$ and $y = x^2$ on $[-1, 2]$



where is the vertical distance is the maximum?

$x + 2 - x^2$ Maximum

$f(x) = -x^2 + x + 2$

Parabola opens down

$f'(x) = 0$ → vertex

C.D.

$f'(x) = -2x + 1$ $\qquad -2x + 1 = 0$

$\boxed{x = \frac{1}{2}}$ → Max. distance is at point $\left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right)$

$f''(x) = -2 < 0$ Max. Point